

Sound waves in (2+1) dimensional holographic magnetic fluids

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ABSTRACT: We use the AdS/CFT correspondence to study propagation of sound waves in strongly coupled (2+1) dimensional conformal magnetic fluids. Our computation provides a nontrivial consistency check of the viscous magneto-hydrodynamics of Hartnoll-Kovtun-Müller-Sachdev to leading order in the external field. Depending on the behavior of the magnetic field in the hydrodynamic limit, we show that it can lead to further attenuation of sound waves in the (2+1) dimensional conformal plasma, or reduce the speed of sound. We present both field theory and dual supergravity descriptions of these phenomena. While to the leading order in momenta the dispersion of the sound waves obtained from the dual supergravity description agrees with the one predicted from field theory, we find a discrepancy at higher order. This suggests that further corrections to HKMS magneto-hydrodynamics are necessary.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence.

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1. Introduction

Gauge theory/string theory correspondence of Maldacena [1–3] proved to be a valuable tool in study dynamical processes in strongly coupled four dimensional gauge theory plasma. One of the most impressive contribution of string theory to the non-equilibrium plasma phenomena was the construction of relativistic conformal hydrodynamics [4, 5], with its emphasis on shortcomings of widely used second order hydrodynamics of Müller-Israel-Stewart (MIS) [6, 7].

It has been realized recently that string theory, and, in particular, dual holographic descriptions of strongly coupled (2+1) dimensional collective dynamics, might shed new light on longstanding problems in condensed matter such as the theory of quantum phase transitions [8], superfluidity [9, 10], and high-temperature superconductivity [11, 12]. Our present paper is largely motivated by these ideas. Specifically, we would like to better understand

conformal viscous hydrodynamics of strongly coupled (2+1) quantum field theories in external fields. Our starting point is the first order dissipative magneto-hydrodynamics proposed by Hartnoll *et.al.* (HKMS) in [8]. Much like four-dimensional MIS relativistic hydrodynamics, it is built on the idea of constructing an entropy current away from equilibrium plus constitutive relations for dissipative currents. The entropy current is constrained to have a positive divergence, an idea originally due to Landau and Lifshitz [13]. We already know that this framework misses some important aspects of relativistic hydrodynamics at second order [4], but it is a well-motivated approximation. In this paper we would like to subject HKMS magneto-hydrodynamics¹ to consistency tests by extracting transport properties from the dispersion relation of the sound waves in strongly coupled $M2$ -brane plasma and comparing them with the transport coefficients obtained from analysis of the current-current correlation functions [15, 16].

Our second motivation is to use the external field as a ‘dial’ to control the (effective) speed of sound waves in strongly coupled plasma. Indeed, the speed of sound waves c_s in conformal plasma and in the absence of external fields is determined by simple scale invariance, $c_s = 1/\sqrt{2}$ in (2+1) conformal fluids, in units where the speed of light is unity. This result is universal in relativistic (2+1) dimensional conformal hydrodynamics without external fields. On the other hand, interesting (2+1) dimensional gapless systems, such as a single atomic layer of graphite [17], have a much smaller speed of propagating sound waves. Thus, if there is any hope of realistically modeling such systems in the context AdS/CFT duality, one needs to understand how to reduce the speed of sound in holographic relativistic plasma. In [18] it was argued that sound waves in 3+1 dimensions are coupled to the magnetic variables of the fluid since the magnetization oscillates with the fluid density. As a result, dispersion relation of the sound waves is affected by the external magnetic fields. We would like to understand here whether such an effect persists in strongly coupled (2+1) dimensional magnetic fluids which admit a dual holographic realization.

In this paper, we explore viscous hydrodynamics of strongly coupled $M2$ -brane plasma in external magnetic field. In the next section we discuss magneto-hydrodynamics from the gauge theory perspective, emphasizing the necessity to appropriately scale the external magnetic field in the hydrodynamic limit. We point out that decoupling of the shear and sound modes in magnetic fluids requires vanishing of the equilibrium charge density (or the corresponding chemical potential). We derive dispersion relation for the propagation of the sound waves, explicitly demonstrating its sensitivity to the background field, similar to what was observed in [18]. In particular, we show that we can reduce the speed of sound by appropriately turning on a magnetic field. In section 3 we analyze magneto-hydrodynamics of dyonic black holes in $AdS_4 \times S^7$ supergravity backgrounds of M-theory. The latter realizes a holographic dual to strongly coupled $M2$ -brane magnetized plasma. We study quasinormal modes of these dyonic black holes, and demonstrate, in parallel with the field-theoretic analysis, that decoupling of the shear and the sound modes occur only for black holes with zero electric charge. We compute dispersion relation of sound quasinormal

¹We postpone detailed analysis of conformal viscous hydrodynamics in external fields to our companion paper [14].

modes, and interpret the results within the framework of HKMS magneto-hydrodynamics. Finally, we conclude in section 4.

2. Gauge theory magneto-hydrodynamics

2.1 Equations of motion and conformal invariance

In this paper, we are interested in hydrodynamic properties of the (2+1)-dimensional theory living on a large number of $M2$ -branes. One can view this theory as the three-dimensional maximally supersymmetric gauge theory near the infrared fixed point. It also admits a holographic description as M-theory on the manifold $AdS_4 \times S^7$ [1]. The state corresponding to the thermal equilibrium is described by the black brane in AdS_4 [19]. The most rigorous way to understand the field theory equations of motion is to use this holographic duality. For a review of the near-boundary holographic analysis of asymptotically AdS space-times see [20].

Let g_{MN} and A_M , $M, N = 0, \dots, 3$ be the metric and the gauge field in AdS_4 .² To construct the dual field theory on the boundary, we have to solve the equations of motion for g_{MN} and A_M with appropriately defined boundary conditions $g_{MN} \rightarrow g_{\mu\nu}^{(0)}$ and $A_M \rightarrow A_\mu^{(0)}$, $\mu, \nu = 0, 1, 2$. According to the AdS/CFT dictionary [21, 19], the boundary correlation functions are encoded in the *renormalized* action $S_{\text{ren}}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]$. In constructing $S_{\text{ren}}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]$ one can perform the integral over the AdS radial coordinate and it becomes a functional on the boundary. This procedure of constructing the renormalized boundary action is known by the name “holographic renormalization”. Then, the boundary stress-energy tensor and the current are defined as

$$\delta S_{\text{ren}}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}] = \int d^3x \sqrt{-g} \left(\frac{1}{2} \langle T_{\mu\nu} \rangle \delta g^{(0)\mu\nu} + \langle J^\mu \rangle \delta A_\mu^{(0)} \right). \quad (2.1)$$

The field theory equations of motion become the consequences of the symmetries of the action S_{ren} . Since S_{ren} is invariant under diffeomorphisms and gauge transformations it follows that we have the boundary conservation equations of the form

$$\begin{aligned} \nabla_\nu T^{\mu\nu} &= F^{\mu\nu} J_\nu, \\ \nabla_\mu J^\mu &= 0, \end{aligned} \quad (2.2)$$

where, to simplify notation, we have removed the brackets $\langle \rangle$. To derive eqs. (2.2) one has to assume that the diffeomorphisms and gauge transformations act non-trivially on the boundary. However, in the bulk there is a special type of diffeomorphism which does not transform the coordinates x^μ on the boundary and whose boundary effect is to Weyl rescale the metric $g_{\mu\nu}^{(0)}$ [22]. Such diffeomorphism can also act non-trivially on the boundary gauge potential [23]. In appendix A we show that the induced transformation on the boundary

²For purposes of the paper, we will specialize to the case of the four-dimensional AdS space with three-dimensional boundary and restrict ourselves to the the metric and gauge bulk fields. Of course, this analysis can be made more general. See [20] and references therein for details.

gauge field is trivial for a massless bulk gauge field. Therefore, under such diffeomorphism, the boundary fields transform as

$$\delta g^{(0)\mu\nu} = -2\sigma(x)g^{(0)\mu\nu}, \quad \delta A_\mu^{(0)} = 0, \quad (2.3)$$

where $\sigma(x)$ is the transformation parameter. See eq. (A.4) in appendix A. The renormalized action S_{ren} has to be invariant under all the bulk diffeomorphisms. The invariance under these transformations implies that³

$$T_\mu{}^\mu = 0. \quad (2.4)$$

Therefore, conformal invariance is unbroken by the presence of the background gauge field. The unbroken conformal invariance can also be understood intuitively if one recalls that the fact that the gauge field goes to a finite x^μ dependent piece $A_\mu^{(0)}(x)$ on the boundary (see appendix A), means that in field theory it represents a marginal deformation [19].

In this paper, we consider hydrodynamics in the presence of the net charge density ρ and the background magnetic field B . For simplicity, we will drop the label “(0)” from the boundary fields in what follows and denote the bulk and the boundary fields by the same letter. However, from the context it will be clear whether the corresponding field belongs to the bulk or to the boundary.

2.2 The first order hydrodynamics

In this subsection, we study hydrodynamic perturbations in the presence of the charge density ρ and the magnetic field B . For our purposes, we can take the boundary metric to be flat and the equations of motion become

$$\begin{aligned} \partial_\nu T^{\mu\nu} &= F^{\mu\nu} J_\nu, \\ \partial_\mu J^\mu &= 0, \end{aligned} \quad (2.5)$$

where $F^{\mu\nu}$ is the field strength corresponding to the background electromagnetic field. One can do the standard decomposition of the stress tensor,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} + \Pi^{\mu\nu}, \quad (2.6)$$

where

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu, \quad \Pi^\mu{}_\nu u^\nu = 0, \quad (2.7)$$

and ϵ and P are the energy density and the pressure respectively. In writing eq. (2.6) we are working in the so-called “energy frame”, and the four velocity of the fluid u^μ is *defined* by the eigenvalue equation $T^\mu{}_\nu u^\nu = -\epsilon u^\mu$. It was argued in [8] that in the presence of the magnetic field the pressure P differs from the usual thermodynamic pressure p by the term $-MB$, where M is the magnetization. The dissipative term $\Pi^{\mu\nu}$ is given by

$$\Pi^{\mu\nu} = -\eta \sigma^{\mu\nu} - \zeta \Delta^{\mu\nu} (\partial_\alpha u^\alpha), \quad (2.8)$$

³The right hand side of this equation has to be supplemented by the conformal anomaly which can also be computed holographically [24]. However, it is not relevant for our discussion since the conformal anomaly vanishes in odd dimensions. In the case of four-dimensional hydrodynamics it is relevant at fourth order as pointed out in [4].

where

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta \partial_\alpha) - \Delta^{\mu\nu} (\partial_\gamma u^\gamma), \quad (2.9)$$

and η and ζ are the shear and bulk viscosity. Note that $\Pi^{\mu\nu}$ is, by definition, zero at local equilibrium.

Now we consider the current J^μ . It is given by

$$J^\mu = \rho u^\mu + \nu^\mu, \quad (2.10)$$

where ν^μ is the dissipative part satisfying $u^\mu \nu_\mu = 0$. The expression for it can be obtained from the argument that the entropy production has to be positive. It was done in [8] and the result is

$$\nu^\mu = \sigma_Q \Delta^{\mu\nu} (-\partial_\nu \mu + F_{\nu\alpha} u^\alpha + \frac{\mu}{T} \partial_\nu T). \quad (2.11)$$

In this expression, T is the temperature, μ is the chemical potential, $F_{\mu\nu}$ is the background field strength and σ_Q is the DC conductivity coefficient. In the case under consideration

$$F_{0i} = 0, \quad i = 1, 2, \quad F_{ij} = \epsilon_{ij} B. \quad (2.12)$$

We would like to study fluctuations around the equilibrium state in which

$$u^\mu = (1, 0, 0), \quad T = \text{const.}, \quad \mu = \text{const.} \quad (2.13)$$

As an independent set of variables we will choose the two components of the velocity $\delta u_1 \equiv \delta u_x$, $\delta u_2 \equiv \delta u_y$ as well as δT and $\delta \mu$. As usual, all perturbations are of the plane-wave form $\exp(-i\omega t + iqy)$. We find that the relevant fluctuations of $T^{\mu\nu}$ are

$$\begin{aligned} \delta T^{tt} &= \delta \epsilon = \left(\frac{\partial \epsilon}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial \epsilon}{\partial T} \right)_\mu \delta T, \\ \delta T^{ti} &= (\epsilon + P) \delta u^i, \\ \delta T^{xy} &= -\eta \partial_y \delta u_x, \\ \delta T^{yy} &= \delta P - (\eta + \zeta) \partial_y \delta u_y = \left(\frac{\partial P}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial P}{\partial T} \right)_\mu \delta T - (\eta + \zeta) \partial_y \delta u_y. \end{aligned} \quad (2.14)$$

Note that since P is different from the thermodynamic pressure, $\left(\frac{\partial P}{\partial \mu} \right)_T$ and $\left(\frac{\partial P}{\partial T} \right)_\mu$ do not coincide with the charge density ρ and the entropy density s . The equalities, however, hold when we set $B = 0$. Similarly, we obtain the following fluctuations of the current

$$\begin{aligned} \delta J^t &= \delta \rho = \left(\frac{\partial \rho}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial \rho}{\partial T} \right)_\mu \delta T, \\ \delta J^x &= \rho \delta u_x + \sigma_Q B \delta u_y, \\ \delta J^y &= \rho \delta u_y + \sigma_Q \left(-\partial_y \delta \mu + \frac{\mu}{T} \partial_y \delta T - B \delta u_x \right). \end{aligned} \quad (2.15)$$

Substituting these expressions into equations of motion (2.5) and performing a Fourier transformation we get the following system of equations

$$\begin{aligned}
0 &= \omega \left(\left(\frac{\partial \epsilon}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial \epsilon}{\partial T} \right)_\mu \delta T \right) - q(\epsilon + P) \delta u_y, \\
0 &= \omega(\epsilon + P) \delta u_y - q \left(\left(\frac{\partial P}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial P}{\partial T} \right)_\mu \delta T \right) + iq^2(\eta + \zeta) \delta u_y + i\sigma_Q B^2 \delta u_y + iB\rho \delta u_x, \\
0 &= \omega(\epsilon + P) \delta u_x - qB\sigma_Q \left(\delta \mu - \frac{\mu}{T} \delta T \right) - iB\rho \delta u_y + i\sigma_Q B^2 \delta u_x + iq^2 \eta \delta u_x, \\
0 &= \omega \left(\left(\frac{\partial \rho}{\partial \mu} \right)_T \delta \mu + \left(\frac{\partial \rho}{\partial T} \right)_\mu \delta T \right) - q\rho \delta u_y + q\sigma_Q B \delta u_x + iq^2 \sigma_Q \left(\delta \mu - \frac{\mu}{T} \delta T \right).
\end{aligned} \tag{2.16}$$

Our aim is to understand how sound and shear modes are modified in the presence of the charge density and the magnetic field. However, eqs. (2.16) are all coupled to each other and it does not seem to be meaningful to ask what happens to the sound and shear modes separately. On the other hand, there is a regime in which these equations simplify and decouple into the two independent pairs. First, we will consider hydrodynamics with no charge density and no chemical potential

$$\rho = 0, \quad \mu = 0. \tag{2.17}$$

In addition, motivated by $M2$ -brane magneto-hydrodynamics, we will set

$$\left(\frac{\partial \rho}{\partial T} \right)_\mu = 0, \quad \left(\frac{\partial \epsilon}{\partial \mu} \right)_T = 0, \quad \left(\frac{\partial \rho}{\partial \mu} \right)_T \neq 0, \quad \left(\frac{\partial \epsilon}{\partial T} \right)_\mu \neq 0. \tag{2.18}$$

Conditions (2.18) are also satisfied at $\rho = 0, \mu = 0$ on the supergravity side which will be studied in the next section. Then, equations (2.16) get separated into the two decoupled pairs. The first pair reads

$$\begin{aligned}
\omega \left(\frac{\partial \epsilon}{\partial T} \right)_\mu \delta T - q(\epsilon + P) \delta u_y &= 0, \\
\omega(\epsilon + P) \delta u_y - q \left(\frac{\partial P}{\partial T} \right)_\mu \delta T + iq^2(\eta + \zeta) \delta u_y + i\sigma_Q B^2 \delta u_y &= 0.
\end{aligned} \tag{2.19}$$

If we set $B = 0$, these two equations describe the sound mode with dispersion relation

$$\omega = \pm c_s q - iq^2 \frac{\eta + \zeta}{\epsilon + P}, \tag{2.20}$$

where the speed of sound is defined, as usual, as $c_s^2 = \frac{\partial P}{\partial \epsilon}$. We will refer to this pair of equation as to the sound mode equations. The second decoupled pair of equations becomes

$$\begin{aligned}
\omega(\epsilon + P) \delta u_x - qB\sigma_Q \delta \mu + i\sigma_Q B^2 \delta u_x + iq^2 \eta \delta u_x &= 0, \\
\omega \left(\frac{\partial \rho}{\partial \mu} \right)_T \delta \mu + q\sigma_Q B \delta u_x + iq^2 \sigma_Q \delta \mu &= 0.
\end{aligned} \tag{2.21}$$

In the absence of the magnetic field these two equations describe a shear perturbation δu_x with dispersion relation

$$\omega = -iq^2 \frac{\eta}{\epsilon + P}, \quad (2.22)$$

and a diffusive mode $\delta\mu$ with dispersion relation

$$\omega = -iq^2 \frac{\sigma_Q}{\left(\frac{\partial\rho}{\partial\mu}\right)_T}. \quad (2.23)$$

We will refer to these equations as to the shear mode equations.

In the presence of the magnetic field all these solutions (2.20), (2.22) and (2.23) disappear. From the sound mode equations we obtain one constant solution

$$\omega = -i \frac{\sigma_Q B^2}{\epsilon + P} + \mathcal{O}(q^2), \quad (2.24)$$

and a diffusive mode

$$\omega = -iq^2 c_s^2 \frac{\epsilon + P}{\sigma_Q B^2} + \mathcal{O}(q^4). \quad (2.25)$$

From the shear mode equations we also obtain a constant mode (2.24) and a subdiffusive mode

$$\omega = -iq^4 \frac{\eta}{B^2 \left(\frac{\partial\rho}{\partial\mu}\right)_T} + \mathcal{O}(q^6). \quad (2.26)$$

In this analysis it has been assumed that the magnetic field B is held fixed in the hydrodynamic limit.

However, it is not clear whether the solutions (2.25) and (2.26) can be trusted. The first hint that the regime of constant B in the hydrodynamic limit might not be well-defined comes from inspecting the B -dependent term in eq. (2.11). The conductivity coefficient σ_Q is of order the free mean path ℓ . Therefore, at $B = 0$ each term in (2.11) is of order $\frac{\ell}{L}$ where L is the scale over which the derivatives vary. However, the term with $F_{\nu\alpha}$ at constant B is not of this order since this term does not contain derivatives. This means that this term is not small in the hydrodynamic limit $\frac{\ell}{L} \ll 1$. It also follows that the limit of small B does not commute with the hydrodynamic limit. In other words, B cannot be thought of as being a small perturbation and one can worry that the hydrodynamic analysis in this case is unstable under higher order corrections. Let us now present a more quantitative reason why the solutions given above might not be reliable. From equations (2.19) it follows that if $\omega \sim q^2$ we obtain

$$\frac{\delta T}{\delta u_y} \sim \frac{1}{q}. \quad (2.27)$$

Thus, assuming that the amplitude δu_y is fixed and of order unity in the hydrodynamic limit, we find that the amplitude of δT is infinitely large. This means that the terms which are naively of higher order because they are suppressed by higher power of ω and q can, in fact, modify hydrodynamics at lower order because of the large amplitude.

Hence, it is more natural to study magnetic fields which vanish in the hydrodynamic limit. That is, we consider B which scales as

$$B = bq^p, \quad p > 0, \quad (2.28)$$

with b held fixed. An interesting observation is that choosing different values of p we can probe hydrodynamics in different regimes. In this paper, we will concentrate on the sound waves for $p = 1$ and $p = 1/2$. Below we will present the field theory results and in the next section, we will study the holographic dual description.

Similar analysis can also be performed for the shear modes. We will not do it in the present paper.

2.3 Sound waves in magnetic field

If the magnetic field vanishes in the hydrodynamic limit, to leading order in q we can consider the various transport coefficients and susceptibilities evaluated at $B = 0$. This, of course, is consistent if we are interested in the first order hydrodynamics. If one wishes to go to the second (or higher) order one has to keep in mind that there will be corrections not only from the term which are of higher order in derivatives but also from the possible $B = bq^p$ -dependence of the transport coefficients. Thus, the shear viscosity η and the conductivity σ_Q can be taken to be equal to their values at $B = 0$. Since, as shown in subsection 2.1, our theory is conformally invariant the bulk viscosity ζ vanishes. Furthermore, the pressure P in this case becomes the usual thermodynamic pressure p . Therefore, we have

$$\left(\frac{\partial P}{\partial T}\right)_\mu = \left(\frac{\partial p}{\partial T}\right)_\mu = s. \quad (2.29)$$

In addition, we have another well-known relation

$$\delta\epsilon = c_s^2 \delta P, \quad (2.30)$$

where $c_s^2 = 1/2$. With these simplifications, we have the following sound mode equations

$$\begin{aligned} \omega\delta\epsilon - q(\epsilon + P)\delta u_y &= 0, \\ \omega(\epsilon + P)\delta u_y - \frac{1}{2}q\delta\epsilon + iq^2\eta\delta u_y + i\sigma_Q B^2\delta u_y &= 0. \end{aligned} \quad (2.31)$$

We would like to study solutions to these equations when B scales as bq and $bq^{1/2}$.

2.3.1 The regime $B = bq$

Substituting $B = bq$ into eqs. (2.31) we obtain

$$\begin{aligned} \omega\delta\epsilon - q(\epsilon + P)\delta u_y &= 0, \\ \omega(\epsilon + P)\delta u_y - \frac{1}{2}q\delta\epsilon + iq^2(\eta + \sigma_Q b^2)\delta u_y &= 0. \end{aligned} \quad (2.32)$$

We see that the effect of the magnetic field is to shift the shear viscosity by the amount $\sigma_Q b^2$. This means that the modified dispersion relation is

$$\omega = \pm \frac{1}{\sqrt{2}}q - \frac{iq^2}{2} \frac{\eta + \sigma_Q b^2}{\epsilon + P}. \quad (2.33)$$

We obtain a sound wave whose speed is $1/\sqrt{2}$ of the speed of light and with modified attenuation.

2.3.2 The regime $B = bq^{1/2}$

Substituting $B = bq^{1/2}$ into eq. (2.31) we obtain

$$\begin{aligned}\omega\delta\epsilon - q(\epsilon + P)\delta u_y &= 0, \\ \omega(\epsilon + P)\delta u_y - \frac{1}{2}q\delta\epsilon + iq^2\eta\delta u_y + i\sigma_Q b^2 q\delta u_y &= 0,\end{aligned}\tag{2.34}$$

and the corresponding characteristic equation becomes

$$\omega^2(\epsilon + P) + i\omega(\eta q^2 + \sigma_Q b^2 q) - \frac{1}{2}q^2(\epsilon + P) = 0.\tag{2.35}$$

Note that if we take σ_Q and $(\epsilon + P)$ to be equal to their values at $B = 0$ we cannot trust the ωq^2 term in this equation since it will be modified due to B^2 dependence of σ_Q and $(\epsilon + P)$. Then the solution becomes the following sound wave

$$\begin{aligned}\omega_{1,2} &= -i\frac{\sigma_Q b^2}{2(\epsilon + P)}q \pm \frac{q}{\sqrt{2}}\sqrt{1 - \frac{\sigma_Q^2 b^4}{2(\epsilon + P)^2}} \\ &= -i\frac{\sigma_Q b^2}{2(\epsilon + P)}q \pm \frac{q}{\sqrt{2}}\left(1 - \frac{\sigma_Q^2 b^4}{4(\epsilon + P)^2} + \mathcal{O}(b^6)\right).\end{aligned}\tag{2.36}$$

Note that the speed of this sound wave is different from $1/\sqrt{2}$ and is decreased in presence of the magnetic field.

Now let us compute corrections of order q^2 to the dispersion relation. Going back to eqs. (2.19) we see that to obtain all necessary terms to the given order we need to expand $(\epsilon + P)$, $\frac{\partial\epsilon}{\partial T}$, $\frac{\partial P}{\partial T}$, σ_Q to next-to-leading order in $B^2 = b^2 q$. Note that $(\eta + \zeta)$ already multiplies q^2 and, hence, can be taken in the limit of the zero magnetic field. Furthermore, from subsection 2.1 we know that the theory is conformally invariant. Hence,

$$P = \frac{\epsilon}{2},\tag{2.37}$$

where, as we have explained before, $P = p - MB$, with p being the thermodynamic pressure and M being the magnetization. Then it follows that eq. (2.35) still holds but we have to expand $(\epsilon + P)$ and σ_Q to next-to-leading order in $B^2 = b^2 q$. Let us denote

$$\begin{aligned}\epsilon + P &= E_0 + qb^2 E_1, \\ \sigma_Q &= \sigma_0 + qb^2 \sigma_1.\end{aligned}\tag{2.38}$$

For simplicity, we will work to order b^4 . Then we obtain the following solutions to (2.35)

$$\begin{aligned}\omega_{1,2} &= -\frac{iq}{2E_0}\left(\frac{\sigma_0 b^2}{E_0} + \eta q + \left(\sigma_1 - \frac{E_1}{E_0}\right)b^4 q\right) \\ &\quad \pm \frac{q}{\sqrt{2}}\left(1 - \frac{\sigma_0^2 b^4}{4E_0^2} - \frac{\sigma_0 \eta b^2}{2E_0^2} q\right).\end{aligned}\tag{2.39}$$

This finishes our field theory consideration. Now we are going to move to the super-gravity side. Our aim will be to reproduce the solutions discussed in this section.

3. Supergravity magneto-hydrodynamics

3.1 Effective action and dyonic black hole geometry

The effective four-dimensional bulk action describing supergravity dual to $M2$ -brane plasma in the external field is given by [15]⁴

$$S_4 = \frac{1}{g^2} \int dx^4 \sqrt{-g} \left[-\frac{1}{4}R + \frac{1}{4}F_{MN}F^{MN} - \frac{3}{2} \right], \quad (3.1)$$

where

$$\frac{1}{g^2} = \frac{\sqrt{2}N^{3/2}}{6\pi}, \quad (3.2)$$

and g is the bulk coupling constant. From eq. (3.1) we obtain the following equations of motion

$$\begin{aligned} R_{MN} &= 2F_{ML}F_N^L - \frac{1}{2}g_{MN}F_{LP}F^{LP} - 3g_{MN}, \\ \nabla_M F^{MN} &= 0. \end{aligned} \quad (3.3)$$

According to the AdS/CFT dictionary, the equilibrium state of magneto-hydrodynamics is described by dyonic black hole geometry whose Hawking temperature is identified with the plasma temperature on the field theory side. A dyonic black hole in AdS_4 with planar horizon is given by the following solution to eqs. (3.3) [15]

$$\begin{aligned} ds_4^2 &\equiv -c_1(r)^2 dt^2 + c_2(r)^2 [dx^2 + dy^2] + c_3(r)^2 dr^2 \\ &= \frac{\alpha^2}{r^2} [-f(r)dt^2 + dx^2 + dy^2] + \frac{1}{r^2} \frac{dr^2}{f(r)}, \\ F &= h\alpha^2 dx \wedge dy + q\alpha dr \wedge dt, \\ f(r) &= 1 + (h^2 + q^2)r^4 - (1 + h^2 + q^2)r^3, \end{aligned} \quad (3.4)$$

where h, q, α are constants related to the field theory quantities as follows [15]

$$B = h\alpha^2, \quad \mu = -q\alpha, \quad \frac{4\pi T}{\alpha} = 3 - \frac{\mu^2}{\alpha^2} - \frac{B^2}{\alpha^4}, \quad (3.5)$$

where B is the external magnetic field of the equilibrium $M2$ -brane plasma, μ is its chemical potential, and T is the plasma temperature.

Let us now review thermodynamics of dyonic black holes. It has been studied extensively in [15] to which we refer for additional details. In the grand canonical ensemble the thermodynamic potential is given by

$$\Omega = -V_2 p = V_2 \frac{1}{g^2} \frac{\alpha^3}{4} \left(-1 - \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4} \right), \quad (3.6)$$

where V_2 is area of the (x, y) -plane and p is the thermodynamic pressure. Just like on the field theory side, as the independent quantities we consider α related to the temperature

⁴For simplicity, we set the radius of AdS_4 to unity.

by eq. (3.5), the chemical potential μ and the magnetic field B . In terms of these variables we can compute the energy ϵ , the entropy s and the electric charge ρ per unit area. One obtains [15]

$$\epsilon = \frac{1}{g^2} \frac{\alpha^3}{2} \left(1 + \frac{\mu^2}{\alpha^2} + \frac{B^2}{\alpha^4} \right). \quad (3.7)$$

Just like on the field theory side, it coincides with the temporal component $\langle T_{00} \rangle$ of the stress-energy tensor. Furthermore, the entropy density is given by

$$s = \frac{\pi}{g^2} \alpha^2. \quad (3.8)$$

Finally, the charge density is

$$\rho = \frac{1}{g^2} \alpha \mu. \quad (3.9)$$

In addition, we introduce the magnetization per unit area

$$M = -\frac{1}{V_2} \left(\frac{\partial \Omega}{\partial B} \right)_{T, \mu} = -\frac{1}{g^2} \frac{B}{\alpha}. \quad (3.10)$$

In the presence of the magnetic field, the thermodynamic pressure p is different from $\langle T_{xx} \rangle$ by the term proportional to the magnetization. Just like on the field theory side, we introduce the pressure P as $\langle T_{xx} \rangle$ which equals to

$$P = p - MB. \quad (3.11)$$

It is easy to check that

$$P = \frac{\epsilon}{2}. \quad (3.12)$$

Notice that the trace of the stress-energy tensor

$$\langle T_\nu^\nu \rangle = 0 \quad (3.13)$$

vanishes, implying unbroken scale invariance.

In parallel to the field theory discussion in the previous section we study sound quasi-normal modes of the *magnetically* charged black hole, i.e., we set

$$q = 0 \quad \iff \quad \mu = 0. \quad (3.14)$$

Then from eqs. (3.6)-(3.12) we obtain

$$\begin{aligned} \left(\frac{\partial \epsilon}{\partial T} \right)_\mu &= \frac{1}{g^2} \frac{2\alpha^2 \pi (3\alpha^4 - B^2)}{3(\alpha^4 + B^2)}, & \left(\frac{\partial \epsilon}{\partial \mu} \right)_T &= 0, \\ \left(\frac{\partial \rho}{\partial T} \right)_\mu &= 0, & \left(\frac{\partial \rho}{\partial \mu} \right)_T &= \frac{1}{g^2} \alpha. \end{aligned} \quad (3.15)$$

This is in exact agreement with our field theory conditions (2.18).

3.2 Fluctuations

Now we study fluctuations in the background geometry

$$\begin{aligned} g_{MN} &\rightarrow g_{MN} + h_{MN}, \\ A_M &\rightarrow A_M + a_M, \end{aligned} \tag{3.16}$$

where g_{MN} and A_M ($dA = F$) are the black brane background configuration (3.4), and $\{h_{MN}, a_M\}$ are the fluctuations. To proceed, it is convenient to choose the gauge

$$h_{tr} = h_{xr} = h_{yr} = h_{rr} = 0, \quad a_r = 0. \tag{3.17}$$

To be consistent with the field theory side, we will take all the fluctuations to depend only on (t, y, r) , i.e., we have a \mathbb{Z}_2 parity symmetry along the x -axis. Strictly speaking, for this parity to be a symmetry the reflection of the x coordinate must be accompanied by the change of the B -field orientation:

$$\mathbb{Z}_2 : \quad x \rightarrow -x \quad \& \quad h \rightarrow -h. \tag{3.18}$$

At a linearized level, and for the vanishing chemical potential $\mu = 0$, we find that the following sets of fluctuations decouple from each other

$$\begin{aligned} \mathbb{Z}_2 - \text{even} : & \quad \{h_{tt}, h_{ty}, h_{xx}, h_{yy}; a_x\}, \\ \mathbb{Z}_2 - \text{odd} : & \quad \{h_{tx}, h_{xy}; a_t, a_y\}. \end{aligned} \tag{3.19}$$

Notice that naively the gauge potential fluctuation a_x is parity-odd, while the other two components are parity-even. This is misleading, as it will turn out that $a_M \propto h$, and thus a_x is parity-even, while $\{a_t, a_y\}$ are parity-odd. The first set of fluctuations is a holographic dual to the sound waves in the $M2$ -brane plasma in the external magnetic field, which is of interest here. The second set describes the shear and diffusive modes. Note that if the electric charge of the black hole q and, hence, the chemical potential is not zero, a_M is a linear combination of the terms some of which are proportional to h and some are proportional to q . Therefore, in this case a_M does not have a definite sign under parity (3.18). As the result, the two sets of fluctuations in eq. (3.19) no longer decouple. To say it differently, if both h and q are non-zero, the electromagnetic background (3.4) is not an eigenstate of the spacial parity $x \rightarrow -x$. However, the decoupling of the sound and shear fluctuations requires that it be an eigenstate. This is in complete agreement with our field theory analysis. Eqs. (2.16) decouple into the two separate pairs of equations only if we set $\mu = 0$, $\rho = 0$.

Let us introduce

$$\begin{aligned} h_{tt} &= c_1(r)^2 \hat{h}_{tt} = e^{-i\omega t + iqy} c_1(r)^2 H_{tt}, \\ h_{ty} &= c_2(r)^2 \hat{h}_{ty} = e^{-i\omega t + iqy} c_2(r)^2 H_{ty}, \\ h_{xx} &= c_2(r)^2 \hat{h}_{xx} = e^{-i\omega t + iqy} c_2(r)^2 H_{xx}, \\ h_{yy} &= c_2(r)^2 \hat{h}_{yy} = e^{-i\omega t + iqy} c_2(r)^2 H_{yy}, \\ a_x &= i e^{-i\omega t + iqy} \hat{a}_x, \end{aligned} \tag{3.20}$$

where $\{H_{tt}, H_{ty}, H_{xx}, H_{yy}, \hat{a}_x\}$ are functions of the radial coordinate only and $c_1(r)$ and $c_2(r)$ are defined in eq. (3.4). Expanding at a linearized level eqs. (3.3) using eqs. (3.16) and eqs. (3.20) we find the following coupled system of ODE's

$$0 = H''_{tt} + H'_{tt} \left[\ln \frac{c_1^2 c_2}{c_3} \right]' + \frac{1}{2} [H_{xx} + H_{yy}]' \left[\ln \frac{c_2}{c_1} \right]' - \frac{c_3^2}{2c_1^2} \left(q^2 \frac{c_1^2}{c_2^2} (H_{tt} + H_{xx}) \right. \\ \left. + \omega^2 (H_{xx} + H_{yy}) + 2\omega q H_{ty} \right) - 3 \frac{c_3^2}{c_2^4} h^2 \alpha^4 (H_{xx} + H_{yy}) + 6 \frac{c_3^2}{c_2^4} h \alpha^2 q \hat{a}_x, \quad (3.21)$$

$$0 = H''_{ty} + H'_{ty} \left[\ln \frac{c_2^4}{c_1 c_3} \right]' + \frac{c_3^2}{c_2^2} \omega q H_{xx} - 4 \frac{c_3^2}{c_2^4} h \alpha^2 (h \alpha^2 H_{ty} + \omega \hat{a}_x), \quad (3.22)$$

$$0 = H''_{xx} + \frac{1}{2} H'_{xx} \left[\ln \frac{c_1^5 c_2}{c_3^2} \right]' + \frac{1}{2} H'_{yy} \left[\ln \frac{c_2}{c_1} \right]' + \frac{c_3^2}{2c_1^2} \left(\omega^2 (H_{xx} - H_{yy}) - q^2 \frac{c_1^2}{c_2^2} (H_{tt} + H_{xx}) \right. \\ \left. - 2\omega q H_{ty} \right) - \frac{c_3^2}{c_2^4} h^2 \alpha^4 (H_{xx} + H_{yy}) + 2 \frac{c_3^2}{c_2^4} h \alpha^2 q \hat{a}_x, \quad (3.23)$$

$$0 = H''_{yy} + \frac{1}{2} H'_{yy} \left[\ln \frac{c_1^5 c_2}{c_3^2} \right]' + \frac{1}{2} H'_{xx} \left[\ln \frac{c_2}{c_1} \right]' + \frac{c_3^2}{2c_1^2} \left(\omega^2 (H_{yy} - H_{xx}) + q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{xx}) \right. \\ \left. + 2\omega q H_{ty} \right) - \frac{c_3^2}{c_2^4} h^2 \alpha^4 (H_{xx} + H_{yy}) + 2 \frac{c_3^2}{c_2^4} h \alpha^2 q \hat{a}_x, \quad (3.24)$$

$$0 = \hat{a}_x'' + \hat{a}_x' \left[\ln \frac{c_1}{c_3} \right]' + \frac{c_3^2}{c_1^2} \hat{a}_x \left(\omega^2 - \frac{c_1^2}{c_2^2} q^2 \right) + \frac{c_3^2}{2c_2^2} h \alpha^2 \left(q (H_{tt} + H_{xx} + H_{yy}) \right. \\ \left. + 2\omega \frac{c_2^2}{c_1^2} H_{ty} \right). \quad (3.25)$$

The number of the second order equations, of course, coincides with the number of the independent fluctuations. Additionally, there are three first order constraints associated with the (partially) fixed diffeomorphism invariance

$$0 = \omega \left([H_{xx} + H_{yy}]' + \left[\ln \frac{c_2}{c_1} \right]' (H_{xx} + H_{yy}) \right) + q \left(H'_{ty} + 2 \left[\ln \frac{c_2}{c_1} \right]' H_{ty} \right), \quad (3.26)$$

$$0 = q \left([H_{tt} - H_{xx}]' - \left[\ln \frac{c_2}{c_1} \right]' H_{tt} \right) + \frac{c_2^2}{c_1^2} \omega H'_{ty} + 4 h \alpha^2 \frac{\hat{a}_x'}{c_2^2}, \quad (3.27)$$

$$0 = [\ln c_1 c_2]' [H_{xx} + H_{yy}]' - [\ln c_2^2]' H'_{tt} + \frac{c_3^2}{c_1^2} \left(\omega^2 (H_{xx} + H_{yy}) + 2\omega q H_{ty} \right. \\ \left. + q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{xx}) \right) - 2 \frac{c_3^2}{c_2^4} h^2 \alpha^4 (H_{xx} + H_{yy}) + 4 \frac{c_3^2}{c_2^4} h \alpha^2 q \hat{a}_x. \quad (3.28)$$

The constraints are just the Einstein's equations obtained by varying the action with respect to the pure gauge metric components h_{tr}, h_{yr} and h_{rr} . We explicitly verified that eqs. (3.21)-(3.25) are consistent with the constraints (3.26)-(3.28).

Now we introduce the fluctuations invariant under the residual diffeomorphisms and gauge transformations preserving the gauge (3.17). Since we have five second-order equations and three constraints there must be two gauge invariant fluctuations. We find them

to be

$$\begin{aligned}
 Z_H &= 4\frac{q}{\omega} H_{ty} + 2 H_{yy} - 2H_{xx} \left(1 - \frac{q^2 c'_1 c_1}{\omega^2 c'_2 c_2} \right) + 2\frac{q^2 c_1^2}{\omega^2 c_2^2} H_{tt}, \\
 Z_A &= \hat{a}_x + \frac{1}{2q} h\alpha^2 (H_{xx} - H_{yy}).
 \end{aligned}
 \tag{3.29}$$

Then from eqs. (3.21)-(3.25) and (3.26)-(3.28) we obtain two decoupled (gauge invariant) equations of motion for Z_H and Z_A ⁵

$$0 = A_H Z_H'' + B_H Z_H' + C_H Z_H + D_H Z_A' + E_H Z_A, \tag{3.30}$$

$$0 = A_A Z_A'' + B_A Z_A' + C_A Z_A + D_A Z_H' + E_A Z_H. \tag{3.31}$$

The connection coefficients $\{A_H, \dots, E_A\}$ can we computed from (3.21)-(3.25), (3.26)-(3.28) and (3.29) using explicit expressions for the c_i 's, see (3.4). Since these coefficients are very cumbersome we will not present them in the paper.⁶ In the next subsection, we will present the explicit form of the equations (3.30) and (3.31) in the limit of small ω and q . This will be sufficient for our purposes.

3.3 Boundary conditions, hydrodynamic limit and the sound wave dispersion relation

3.3.1 Boundary conditions

According to the general prescription [25, 26], in order to obtain the dispersion relation (poles in the retarded Green's functions) we have to impose the following boundary conditions on the gauge invariant fluctuations $\{Z_H, Z_A\}$.

- $\{Z_H, Z_A\}$ must have incoming wave boundary conditions near the horizon (as $r \rightarrow 1$);
- $\{Z_H, Z_A\}$ must be normalizable near the boundary (as $r \rightarrow 0$).

The second condition is imposed because coefficients of non-normalizable solutions appear as poles in the retarded Green's functions. Thus, setting them to zero will produce the dispersion relation. See [26] for details.

Since the solution of interest is an incoming wave at the horizon it has the following general structure

$$Z_H(r) = [f(r)]^{\beta_1} z_H(r), \quad Z_A(r) = [f(r)]^{\beta_2} z_A(r), \tag{3.32}$$

where $f(r)$ is given by eq. (3.4) and the functions $z_H(r)$ and $z_A(r)$ are non-singular at the horizon. Then denoting

$$\lim_{r \rightarrow 1_-} z_H(r) \rightarrow z_H^{(0)} \neq 0, \quad \lim_{r \rightarrow 1_-} z_A(r) \rightarrow z_A^{(0)} \neq 0, \tag{3.33}$$

⁵To achieve the decoupling one has to use the background equations of motion, i.e., the decoupling occurs only on-shell.

⁶The precise form of the equations (3.30) and (3.31) is available from the authors upon request.

we find from eq. (3.30) and eq. (3.31) that as $x \equiv 1 - r \rightarrow 0_+$ the following two equations must be satisfied

$$\begin{aligned}
0 = & \alpha \mathfrak{w}^2 (h^2 - 3)^4 (h^2 \mathfrak{q}^2 - 3\mathfrak{q}^2 + 4\mathfrak{w}^2)^2 (4\beta_1^2 + \mathfrak{w}^2) z_H^{(0)} \times (1 + \mathcal{O}(x)) \\
& + 8(h^2 - 3)^4 (h^2 \mathfrak{q}^2 - 3\mathfrak{q}^2 + 4\mathfrak{w}^2)^2 (-h^2 \mathfrak{q}^2 + 3\mathfrak{q}^2 + 16\beta_2 - 4\mathfrak{w}^2) \mathfrak{q} h x ((3 - h^2)x)^{\beta_2 - \beta_1} z_A^{(0)} \\
& \times (1 + \mathcal{O}(x)) ,
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
0 = & 2\alpha h \mathfrak{w}^2 (h^2 - 3)^3 (h^2 \mathfrak{q}^2 - 3\mathfrak{q}^2 + 4\mathfrak{w}^2) (-h^2 \mathfrak{q}^2 + 3\mathfrak{q}^2 + 8\beta_1) x z_H^{(0)} \times (1 + \mathcal{O}(x)) \\
& + \mathfrak{q} (h^2 - 3)^4 (h^2 \mathfrak{q}^2 - 3\mathfrak{q}^2 + 4\mathfrak{w}^2)^2 (\mathfrak{w}^2 + 4\beta_2^2) ((3 - h^2)x)^{\beta_2 - \beta_1} z_A^{(0)} \times (1 + \mathcal{O}(x)) ,
\end{aligned} \tag{3.35}$$

where we have defined $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = q/(2\pi T)$. From eqs. (3.34) and (3.35), the existence of a nontrivial solution to (3.30) and (3.31) with incoming wave boundary conditions implies

$$\begin{aligned}
0 = & \alpha \mathfrak{w}^2 \mathfrak{q} ((3 - h^2)x)^{\beta_2 - \beta_1} (3 - h^2)^8 (\mathfrak{q}^2 h^2 + 4\mathfrak{w}^2 - 3\mathfrak{q}^2)^4 \\
& \times \left\{ (\mathfrak{w}^2 + 4\beta_2^2)(\mathfrak{w}^2 + 4\beta_1^2) + \mathcal{O}(x) \right\} .
\end{aligned} \tag{3.36}$$

Eq. (3.36) suggests the following critical exponents:

$$\beta_1 = \beta_2 = -i \frac{\mathfrak{w}}{2} . \tag{3.37}$$

A detailed supergravity analysis of the solutions confirms that (3.37) is indeed the correct choice.

Since (3.30) and (3.31) are second order ODE's, each of them has two independent solutions. Analyzing (3.30), (3.31) as $r \rightarrow 0$ shows that normalizability of z_H implies that

$$z_H(r) = \mathcal{O}(r^3) \quad \text{as} \quad r \rightarrow 0 . \tag{3.38}$$

While both solutions of z_A are normalizable as $r \rightarrow 0$, requiring a fixed background magnetic field at the boundary implies that

$$z_A(r) = \mathcal{O}(r) \quad \text{as} \quad r \rightarrow 0 . \tag{3.39}$$

3.3.2 Hydrodynamic limit with $h \propto \mathfrak{q}$

The aim of this subsection is to find the holographic solution dual to hydrodynamics with the magnetic field $B \sim q$. For this we have to study physical fluctuation equations (3.30) and (3.31), subject to the boundary conditions (3.37) and (3.38), (3.39) in the hydrodynamic approximation, $\mathfrak{w} \rightarrow 0$, $\mathfrak{q} \rightarrow 0$ with $\frac{\mathfrak{w}}{\mathfrak{q}}$ and $\frac{h}{\mathfrak{q}}$ kept constant.

To facilitate the hydrodynamic scaling we parametrize

$$h \equiv |\mathfrak{q}|H . \tag{3.40}$$

Furthermore, we parametrize the sound quasinormal mode dispersion relation as follows

$$\mathfrak{w} = c_s \mathfrak{q} - \frac{i}{2} \left(\Gamma \mathfrak{q}^2 + \Gamma_h h^2 \right) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}h), \quad (3.41)$$

with $\{c_s, \Gamma, \Gamma_h\}$ kept fixed in the hydrodynamic scaling. Without loss of generality we can assume that Γ does not depend on H^2 , while $\Gamma_h = \Gamma_h(H^2)$. We will look for a solution as a power expansion in \mathfrak{q} and introduce

$$z_H = z_{H,0} + i \mathfrak{q} z_{H,1} + \mathcal{O}(\mathfrak{q}^2), \quad z_A = h (z_{A,0} + i \mathfrak{q} z_{A,1} + \mathcal{O}(\mathfrak{q}^2)). \quad (3.42)$$

Notice the $\mathcal{O}(h)$ scaling of z_A in the hydrodynamic limit. This is motivated by the field theory analysis. If $\rho = 0$, the x components of the current in (2.15) is proportional to the magnetic field. Since the equations for z_H and z_A are homogeneous we can rescale $z_H(r)$ and $z_A(r)$ to make $z_H(r)$ be equal to unity on the horizon. It is convenient to make a choice that the leading solution $z_{H,0}$ equals to unity and all the subleading contributions $z_{H,1}(r), \dots$ vanish on the horizon. That is, we impose that

$$\lim_{r \rightarrow 1_-} z_H(r) = 1, \quad \lim_{r \rightarrow 1_-} z_{H,0} = 1, \quad \lim_{r \rightarrow 1_-} z_{H,1} = 0. \quad (3.43)$$

To leading order in the hydrodynamic approximation, and subject to the appropriate boundary conditions, we find from (3.30) and (3.31) that

$$\begin{aligned} 0 = z''_{H,0} &- \left((9(4c_s^2 + r^3 - 4)^2 + 64r^2(r-1)(r^2+r+1)(4c_s^2-3)H^2) \right. \\ &\times (r^2+r+1)(r-1)r \left. \right)^{-1} \left(9(4-4c_s^2-r^3)(4r^3c_s^2+8c_s^2-8+4r^3-5r^6) \right. \\ &+ 128r^2(r-1)^2(r^2+r+1)^2(4c_s^2-3)H^2 \left. \right) z'_{H,0} + \left((9(4c_s^2+r^3-4)^2 \right. \\ &+ 64r^2(r-1)(r^2+r+1)(4c_s^2-3)H^2)(r^2+r+1)(r-1) \left. \right)^{-1} \\ &\times 81(4c_s^2+r^3-4)r^4 z_{H,0}, \end{aligned} \quad (3.44)$$

and

$$0 = z'_{H,0} + \frac{3r^2}{4c_s^2 - 2 - r^3} z_{H,0}. \quad (3.45)$$

Solving (3.45) subject to the boundary conditions (3.43) implies that

$$z_{H,0} = r^3, \quad c_s = \pm \frac{1}{\sqrt{2}}. \quad (3.46)$$

Given (3.46), it is straightforward to verify that (3.44) is satisfied as well. Thus, we have solved the equations of motion to leading order in the hydrodynamic approximation.

Now we move to next-to-leading order. Using (3.46), we find from (3.30) and (3.31) the following two equations

$$\begin{aligned}
0 = z''_{H,1} &- \frac{128r^2(r-1)^2(r^2+r+1)^2H^2 - 9(r^3-2)(5r^6-6r^3+4)}{r\Delta_3(r^2+r+1)(r-1)} z'_{H,1} \\
&+ \frac{576i(r^3-2)H^2r^4}{\alpha\Delta_3} z'_{A,0} - \frac{81(r^3-2)r^4}{\Delta_3(r^2+r+1)(r-1)} z_{H,1} \\
&+ \frac{9\sqrt{2}(2H^2(9\Gamma_h-8r^2)-9+18\Gamma)(r^3-2)r^4}{\Delta_3(r^2+r+1)(r-1)},
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
0 = z''_{A,0} &+ \frac{r(64(r^3-1)(r^3+2)H^2 - 27r(r^3-2)^2)}{\Delta_3(r^3-1)} z'_{A,0} + \frac{6i r^2\alpha}{\Delta_3} z'_{H,1} - \frac{18i r\alpha}{\Delta_3} z_{H,1} \\
&+ \frac{9i r\sqrt{2}\alpha(8\Gamma_h H^2(r^3-1) + 8r^3\Gamma - r^6 - 8\Gamma)}{2\Delta_3(r^3-1)},
\end{aligned} \tag{3.48}$$

where

$$\Delta_3 = 64r^2(r-1)(r^2+r+1)H^2 - 9(r^3-2)^2. \tag{3.49}$$

It is difficult to solve eqs. (3.47) and (3.48) analytically for all values of H^2 . However, for our purposes it is enough to find a solution to order H^2 . Expanding $z_{H,1}$, $z_{A,1}$ and Γ_h perturbatively in H^2 ,

$$\begin{aligned}
z_{H,1} &= z_{H,1,0} + H^2 z_{H,1,1} + \mathcal{O}(H^4), & z_{A,0} &= z_{A,0,0} + H^2 z_{A,0,1} + \mathcal{O}(H^4), \\
\Gamma_h &= \Gamma_{h,0} + \mathcal{O}(H^2),
\end{aligned} \tag{3.50}$$

we find that the solution satisfying the boundary conditions stated in eqs. (3.38), (3.39) and (3.43) is given by

$$\begin{aligned}
z_{H,1,0} &= 0, & z_{H,1,1} &= 0, & z_{A,0,1} &= 0, \\
z_{A,0,0} &= - \frac{i\sqrt{2}\alpha \left(9\ln(r^2+r+1) + 6\sqrt{3}\arctan\left(\frac{2r+1}{\sqrt{3}}\right) - \pi\sqrt{3} \right)}{72}, \\
\Gamma &= \frac{1}{2}, & \Gamma_{h,0} &= \frac{8}{9}.
\end{aligned} \tag{3.51}$$

Here the last line comes from requiring that $z_{H,1}$ vanishes on the boundary.

3.3.3 Comparison with field theory

To summarize the results, we have obtained the following dispersion relation

$$\mathfrak{w} = \pm \frac{1}{\sqrt{2}} \mathfrak{q} - \frac{i}{2} \left(\frac{1}{2} \mathfrak{q}^2 + \frac{8}{9} H^2 \mathfrak{q}^2 \right) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2 H). \tag{3.52}$$

Let us compare it with the field theory counterpart (2.33). For this we will rewrite eq. (3.52) in notation of (2.33). First, we will recall that

$$\mathfrak{w} = \frac{\omega}{2\pi T}, \quad \mathfrak{q} = \frac{q}{2\pi T}. \tag{3.53}$$

Second, we will express the variables (H, α) which are natural to use on the supergravity side in terms of (b, T) . We get

$$H = \frac{2\pi T b}{\alpha^2} = \frac{2\pi T b}{\alpha_0^2} + \mathcal{O}(b^3 q^2), \quad (3.54)$$

where

$$\alpha_0 = \frac{4\pi T}{3}. \quad (3.55)$$

Note that the relation is non-linear because α also depends on $B = bq$. However, for our purposes it is sufficient to take it to leading order. Substituting eqs. (3.53)-(3.55) into eq. (3.52) we obtain (up to higher order terms)

$$\mathfrak{w} = \frac{1}{\sqrt{2}} \mathfrak{q} - \frac{i}{2} \left(\frac{\mathfrak{q}^2}{4\pi T} + \frac{4}{3} \frac{b^2 \mathfrak{q}^2}{\alpha_0^3} \right). \quad (3.56)$$

Comparing it with eq. (2.33), first, we find that

$$\frac{\eta}{\epsilon + P} = \frac{1}{4\pi T}. \quad (3.57)$$

This result agrees with earlier calculations of the shear viscosity η in [27, 28]. Second, we obtain

$$\frac{\sigma_Q}{\epsilon + P} = \frac{4}{3\alpha_0^3}. \quad (3.58)$$

Recalling results for ϵ and P from subsection 3.1 we conclude that

$$\sigma_Q = \frac{1}{g^2}. \quad (3.59)$$

Thus, we reproduced the result for the conductivity coefficient (to leading order in the magnetic field) obtained earlier in [8, 16].

We see that we have a perfect agreement with our field theory analysis and with the earlier calculations of the transport coefficients.

3.3.4 Hydrodynamic limit with $h \propto \sqrt{|\mathfrak{q}|}$

In this subsection, we will study the holographic solution dual to hydrodynamics with the magnetic field $B \sim \sqrt{q}$. For this we will consider the physical fluctuation equations (3.30) and (3.31), subject to the boundary conditions (3.37) and (3.38), (3.39) in the hydrodynamic approximation, $\mathfrak{w} \rightarrow 0$, $\mathfrak{q} \rightarrow 0$ with $\frac{\mathfrak{w}}{\mathfrak{q}}$ and $\frac{h}{\sqrt{|\mathfrak{q}|}}$ kept constant.

The analysis will be similar to the one in subsection 3.3.2. To facilitate the hydrodynamic scaling we parametrize

$$h \equiv \sqrt{|\mathfrak{q}|} H. \quad (3.60)$$

The sound quasinormal mode dispersion relation is parametrized as follows

$$\begin{aligned} \mathfrak{w} &= \Delta h^2 + c_s \mathfrak{q} - \frac{i}{2} \left(\Gamma \mathfrak{q}^2 + \Gamma_h h^4 \right) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2 h^2, \mathfrak{q} h^4, h^6) \\ &= \Gamma_0 \mathfrak{q} + i\Gamma_1 \mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3), \\ \Gamma_0 &\equiv \Delta H^2 + c_s, \\ \Gamma_1 &= -\frac{1}{2} \left(\Gamma + \Gamma_h H^4 \right), \end{aligned} \quad (3.61)$$

with $\{\Delta, c_s, \Gamma, \Gamma_h\}$ kept fixed in the hydrodynamic scaling. We look for a solution as a series in \mathfrak{q} ,

$$z_H = z_{H,0} + i \mathfrak{q} z_{H,1} + \mathfrak{q}^2 z_{H,2} + \mathcal{O}(\mathfrak{q}^3), \quad z_A = h (z_{A,0} + i \mathfrak{q} z_{A,1} + \mathfrak{q}^2 z_{A,2} + \mathcal{O}(\mathfrak{q}^3)), \quad (3.62)$$

and impose (without loss of generality) that

$$\lim_{r \rightarrow 1_-} z_H(r) = 1 \quad \Rightarrow \quad \lim_{r \rightarrow 1_-} z_{H,0} = 1 \ \& \ \lim_{r \rightarrow 1_-} z_{H,1} = \lim_{r \rightarrow 1_-} z_{H,2} = 0. \quad (3.63)$$

To leading order in the hydrodynamic approximation we obtain from (3.30):

$$0 = z''_{H,0} - \frac{2}{r} z'_{H,0}, \quad (3.64)$$

which gives rise to the solution

$$z_{H,0} = r^3. \quad (3.65)$$

Using this solution for $z_{H,0}$, we find from (3.31) the following equation for $z_{A,0}$

$$0 = z''_{A,0} + \frac{r^3 + 2}{(r^3 - 1)r} z'_{A,0} + \frac{9\Gamma_0^2(2\Gamma_0^2 - 1)\alpha}{8(r^3 - 1)(4\Gamma_0^2 - 3)H^2 r}. \quad (3.66)$$

Solving (3.66) subject to the boundary conditions (3.39) gives

$$z_{A,0} = \frac{(2\Gamma_0^2 - 1)\Gamma_0^2\alpha}{32H^2(4\Gamma_0^2 - 3)} \left(\pi\sqrt{3} - 6\sqrt{3} \arctan \frac{1+2r}{\sqrt{3}} - 9 \ln(r^2 + r + 1) \right). \quad (3.67)$$

This finishes our consideration of the equations of motion to leading order. Note that we were not able to determine any of the transport coefficients in the dispersion relation (3.61). Moving on to next-to-leading order in the hydrodynamic approximation we find from (3.30) and (3.31)

$$\begin{aligned} 0 = z''_{H,1} - \frac{2}{r} z'_{H,1} + \frac{9i r^2(r^3 + 4\Gamma_0^2 - 4)}{2\Gamma_0^2\alpha(r^3 - 1)} z'_{A,0} - \frac{i r^2}{32(r^3 - 1)^2(4\Gamma_0^2 - 3)H^2} \left(324 - 81r^3 \right. \\ \left. + 432i (r^3 - 2)H^2 r^2 \Gamma_0 + (162r^3 + 512H^4 r^6 - 972 - 512H^4 r^3)\Gamma_0^2 \right. \\ \left. - 576i (r^3 - 2)H^2 r^2 \Gamma_0^3 + 648\Gamma_0^4 \right), \end{aligned} \quad (3.68)$$

and

$$\begin{aligned} 0 = z''_{A,1} + \frac{r^3 + 2}{r(r^3 - 1)} z'_{A,1} + \frac{3\Gamma_0^2(4\Gamma_0^2 - 2 - r^3)\alpha}{16H^2 r^3(r^3 - 1)(4\Gamma_0^2 - 3)} z'_{H,1} \\ + \frac{9\Gamma_0^2\alpha}{16rH^2(r^3 - 1)(4\Gamma_0^2 - 3)} z_{H,1} + J_{\text{source}} \left[z_{A,0}(r); \{\Gamma_0, \Gamma_1, H\}; r \right], \end{aligned} \quad (3.69)$$

where the source term J_{source} is a linear functional of $z_{A,0}$ also depending on the transport coefficients and the magnetic field $\{\Gamma_0, \Gamma_1, H\}$, as well as explicitly on r .⁷

⁷The explicit expression for J_{source} is extremely lengthy and cumbersome and we find it meaningless to put it in the paper. It is available from the authors upon request.

It is straightforward to analyze the asymptotic solution to (3.68) near the horizon, i.e., as $x \equiv 1 - r \rightarrow 0_+$. For generic values of Γ_0 we find that $z_{H,1}$ has a simple pole and a logarithmic singularity near the horizon. Namely,

$$z_{H,1} = \mathcal{C} \left\{ -\frac{i}{32H^2} \ln x \right\} + \text{finite}, \quad (3.70)$$

where

$$\mathcal{C} = 18\Gamma_0^2 + 16i H^2\Gamma_0 - 9. \quad (3.71)$$

Regularity at the horizon implies that $\mathcal{C} = 0$. It leads to the following solution

$$\Gamma_0 = c_s + \Delta H^2 = -i \frac{4}{9} H^2 \pm \frac{1}{\sqrt{2}} \sqrt{1 - \frac{32}{81} H^4}. \quad (3.72)$$

Without loss of generality, we can assume that c_s is independent of H^2 , while all the H^2 dependence resides in Δ . Hence, we obtain

$$c_s = \pm \frac{1}{\sqrt{2}}, \quad \Delta_{\pm} = -i \frac{4}{9} \mp \frac{1}{H^2 \sqrt{2}} \left(1 - \sqrt{1 - \frac{32}{81} H^4} \right), \quad (3.73)$$

where the signs \pm are correlated in the above expressions. Given (3.73), a nonsingular function $z_{H,1}$ subject to the boundary conditions (3.38), (3.63) is uniquely specified:

$$z_{H,1} = \frac{2i H^2(1-r)(16H^2\Gamma_0 + 9i)}{27(32H^2\Gamma_0 - 9i)} \times \left((1+r+r^2)(12\sqrt{3} \arctan \frac{1+2r}{\sqrt{3}} \right. \\ \left. + 18 \ln(1+r+r^2)) - 2\sqrt{3}\pi - (36 + 2\sqrt{3}\pi)(r+r^2) - 27r^3 \right). \quad (3.74)$$

Now we move on to eq (3.69). Its general solution is of the form

$$z'_{A,1} = \frac{r^2}{r^3 - 1} \left(\int_0^r d\xi \frac{(1 - \xi^3) \hat{J}_{\text{source}}(\xi)}{\xi^2} + \mathcal{C}_{A,1,1} \right) \quad (3.75)$$

where $\mathcal{C}_{A,1,1}$ is an arbitrary integration constant and

$$\hat{J}_{\text{source}}(r) \equiv J_{\text{source}} \left[z_{A,0}(r); \{\Gamma_0, \Gamma_1, H\}; r \right] + \frac{3\Gamma_0^2(4\Gamma_0^2 - 2 - r^3)\alpha}{16H^2r^3(r^3 - 1)(4\Gamma_0^2 - 3)} z'_{H,1} \\ + \frac{9\Gamma_0^2\alpha}{16rH^2(r^3 - 1)(4\Gamma_0^2 - 3)} z_{H,1} \quad (3.76)$$

Since

$$\hat{J}_{\text{source}} \propto \frac{1}{r}, \quad \text{as } r \rightarrow 0_+, \\ \hat{J}_{\text{source}} \propto \frac{1}{x}, \quad \text{as } x = 1 - r \rightarrow 0_+, \quad (3.77)$$

it is clear from (3.75) that for any value of Γ_1 we can adjust the integration constant $\mathcal{C}_{A,1,1}$ to remove singularity of $z_{A,1}$ at the horizon, $x \rightarrow 0_+$; the second constant (obtained from

integrating (3.75)) can be fixed to insure that $z_{A,1}$ vanishes as $r \rightarrow 0$, see (3.39). Thus, we cannot determine Γ_1 at this order. This is not surprising, given that to determine the leading order transport coefficient Γ_0 we had to consider the $\mathcal{O}(q)$ order in the hydrodynamic approximation of z_H .

In order to determine Γ_1 we have to consider $\mathcal{O}(q^2)$ in the hydrodynamic approximation for z_H . The equation of motion for $z_{H,2}$ takes the following form

$$0 = z_{H,2}''' - \frac{2}{r} z_{H,2}' - \frac{9i r^2 (r^3 + 4\Gamma_0^2 - 4)}{2\Gamma_0^2 \alpha (r^3 - 1)} z_{A,1}' + \mathcal{I}_{\text{source}} \left[z_{A,0}, z_{H,1}; \{\Gamma_0, \Gamma_1, H\}; r \right], \quad (3.78)$$

where $\mathcal{I}_{\text{source}}$ is a new source term.⁸ Although we can not solve analytically for $z_{H,2}$, it is straightforward to construct a power series solution, first, for $z_{A,1}$ and then for $z_{H,2}$ near the horizon, $x = 1 - r \rightarrow 0_+$. Much like in (3.70), we find that $z_{H,2}$ is non-singular at the horizon, provided

$$\Gamma_1 = -\frac{1}{4} - \frac{8}{9}H^4 \pm \frac{2iH^2(32H^4 - 45)}{9\sqrt{162 - 64H^4}} \quad (3.79)$$

where \pm sign correlates with the corresponding sign in (3.72). Perturbatively in H^2 ,

$$\Gamma_1 = -\frac{1}{4} \mp \frac{i 5\sqrt{2}}{9}H^2 + \mathcal{O}(H^4). \quad (3.80)$$

3.3.5 Comparison with field theory

To summarize computations performed above, we have obtained the following sound wave

$$\mathfrak{w} = q \left(-\frac{4i}{9}H^2 \pm \frac{1}{\sqrt{2}}\sqrt{1 - \frac{32}{81}H^4} \right) + iq^2 \left(-\frac{1}{4} \mp \frac{i 5\sqrt{2}}{9}H^2 + \mathcal{O}(H^4) \right). \quad (3.81)$$

We would like to compare it with field theory. First, we will consider terms of order q and compare them with the field theory counterpart (2.36). Following the same logic as in subsection 3.3.3 it is straightforward to check that terms of order q are in total agreement with eq. (2.36) if, as before,

$$\frac{\eta}{\epsilon + P} = \frac{1}{4\pi T}, \quad \sigma_Q = \frac{1}{g^2}. \quad (3.82)$$

Thus, we have obtained an agreement with field theory at leading order in q . Note that it holds to all orders in H .

Let us now compare terms of order q^2 in eq. (3.81) with the corresponding field theory prediction (2.39). It is easy to check (it has already been done in subsection 3.3.3) that the results agree for vanishing magnetic field. However, we have a disagreement at order $q^2 H^2$. The field theory result at this order is

$$\mathfrak{w} \sim \mp \sqrt{2} \frac{q^2 H^2}{9}, \quad (3.83)$$

⁸Its explicit expression is too lengthy and cumbersome to be put in the paper. It is available from the authors upon request.

where eqs. (3.82) have been taken into account. On the contrary, the supergravity result is

$$\mathfrak{w} \sim \pm \frac{5\sqrt{2}}{9} \mathfrak{q}^2 H^2. \tag{3.84}$$

We have been unable to identify the source of this disagreement in the framework of HKMS viscous magneto-hydrodynamics. It is natural to expect that introduction of additional transport coefficients would resolve this puzzle. We hope to discuss this issue in more details in [14].

4. Conclusion

In this paper we discussed propagation of the sound waves in magnetic fluids in (2+1) dimensions. We used the strongly coupled $M2$ -brane plasma and the general setting of the holographic gauge theory/string theory duality to test relativistic viscous magneto-hydrodynamics of HKMS [8]. We found that HKMS magneto-hydrodynamics, generalizing the arguments of Landau and Lifshitz [13] in constructing dissipative entropy currents, adequately describes propagation of sound modes to the order in the hydrodynamic limit first sensitive to the external field. There is, however, a disagreement in next order in the hydrodynamic approximation (still in the context of the first-order hydrodynamics). Such a disagreement suggests that additional transport coefficients, beyond those introduced in [8], might be needed to describe dissipation in (2+1) magnetic fluids even to first order in the local velocity gradients.

Finally, it would be interesting to analyze shear modes in (2+1) dimensional strongly coupled magnetic plasma. While the general theorem [29] guarantees that the shear viscosity attains its universal value [30–32], a deeper understanding of the propagation of the shear modes might help in constructing a general theory of relativistic viscous magneto-hydrodynamics.

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A. Bulk diffeomorphisms and boundary Weyl transformations

In this section, we show that the holographic stress-energy tensor is indeed traceless⁹. For the sake of generality, we will work in a bulk of an arbitrary dimension $d + 1$. We write

⁹Up to a possible conformal anomaly, which is known to be zero in odd dimensions.

the metric using the Fefferman-Graham coordinates [33] as follows

$$ds^2 = \frac{1}{r^2} (dr^2 + g_{\mu\nu}(r, x) dx^\mu dx^\nu) . \quad (\text{A.1})$$

The boundary condition on the metric is

$$\lim_{r \rightarrow 0} g_{\mu\nu}(r, x) = g_{\mu\nu}^{(0)} , \quad (\text{A.2})$$

where $g_{\mu\nu}^{(0)}$ is the background metric which the gauge theory is defined. We do not need to worry about the exact nature of the sub-leading terms.

The bulk diffeomorphisms that preserve the form of the metric (A.1) satisfy the Killing's equations

$$\nabla_r \xi_r = 0 , \quad \nabla_{(\mu} \xi_{r)} = 0 . \quad (\text{A.3})$$

We are interested in bulk diffeomorphisms that generate a Weyl transformation in the boundary metric $g_{\mu\nu}^{(0)}$. One can show that such solutions have the asymptotic behavior

$$\xi_r = \frac{\sigma(x)}{r} + \dots , \quad \xi_\mu = -\frac{1}{2} \partial_\mu \sigma(x) + \dots \quad (\text{A.4})$$

It is straightforward to show that the transformation of the boundary metric under the diffeomorphism (A.4) is given by

$$\delta g_{\mu\nu}^{(0)} = \lim_{r \rightarrow 0} (-\mathcal{L}_\xi g_{\mu\nu}) = 2\sigma(x) g_{\mu\nu}^{(0)} . \quad (\text{A.5})$$

In order to compute the transformation of the boundary gauge field, we need to know the asymptotic behavior of the classical solutions of the bulk Maxwell's equations

$$\nabla_M F^{MN} = 0 . \quad (\text{A.6})$$

In the gauge $A_r = 0$, Maxwell's equations take the form

$$\ddot{A}_\mu + (2-d)\dot{A}_\mu + \frac{1}{2} g^{\alpha\beta} \dot{g}_{\alpha\beta} \dot{A}_\mu + g_{\mu\alpha} \dot{g}^{\alpha\beta} \dot{A}_\beta + \frac{r^2}{\sqrt{-g}} g_{\mu\alpha} \partial_\beta \left(\sqrt{-g} g^{\beta\gamma} g^{\alpha\delta} F_{\gamma\delta} \right) = 0 , \quad (\text{A.7})$$

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \dot{A}_\nu \right) = 0 , \quad (\text{A.8})$$

where $\dot{f} = r \partial_r f$. One can then look for asymptotic solutions to these equations in the form $A_\mu \sim r^k A_\mu^{(0)}(x) + \dots$. One finds that the exponent k must satisfy

$$k(2-d+k) = 0 . \quad (\text{A.9})$$

Therefore, the leading term near the boundary is finite ($k=0$)

$$\lim_{r \rightarrow 0} A_\mu(r, x) = A_\mu^{(0)}(x) , \quad (\text{A.10})$$

and we associate it with the boundary gauge field.

For the sake of completeness, let us mention that in the case of a massive bulk gauge field, the r.h.s. of the first Maxwell equation (A.7) equals $m^2 A_\mu$. Therefore the asymptotic behavior of the field is modified to

$$\lim_{r \rightarrow 0} A_\mu(r, x) = r^k A_\mu^{(0)}(x), \quad (\text{A.11})$$

where

$$k = \frac{1}{2} \left[d - 2 - \sqrt{(d-2)^2 + 4m^2} \right]. \quad (\text{A.12})$$

We still identify $A_\mu^{(0)}(x)$ as the background gauge field in the field theory.

We are now ready to study the transformation properties of the bulk gauge field under the diffeomorphisms (A.4). Our gauge is $A_r = 0$. However, one needs to check that the diffeomorphisms (A.4) respect such gauge choice. That is, in general one might need to make a compensating gauge transformation to ensure $\delta A_r = 0$. The leading change in the A_r component is

$$\lim_{r \rightarrow 0} \delta_\xi A_r = \lim_{r \rightarrow 0} (-\mathcal{L}_\xi A_r) = r^{k+1} A_\mu^{(0)} \partial^\mu \sigma(x). \quad (\text{A.13})$$

We see that the compensating gauge transformation must be subleading at the boundary. More precisely, we take $\delta_\lambda A_M = \partial_M \lambda$, and from the requirement $(\delta_\xi + \delta_\lambda) A_r = 0$ we find that

$$\lim_{r \rightarrow 0} \lambda = -\frac{1}{k+2} r^{k+2} A_\mu^{(0)} \partial^\mu \sigma(x). \quad (\text{A.14})$$

Finally, we find that the total asymptotic transformation of the gauge field in the x^μ directions is

$$\lim_{r \rightarrow 0} (\delta_\xi + \delta_\lambda) A_\mu = -k \sigma(x) A_\mu^{(0)}(x) r^k, \quad (\text{A.15})$$

which translates to

$$\delta A_\mu^{(0)} = -k \sigma(x) A_\mu^{(0)}(x). \quad (\text{A.16})$$

Inserting the transformation rules (A.5) and (A.16) into the boundary action (2.1), we obtain the Ward identity

$$\langle T_\mu^\mu \rangle = k \langle J^\mu \rangle A_\mu^{(0)}, \quad (\text{A.17})$$

where k is given in eq. (A.12) for a massive gauge field. For a massless gauge field ($k = 0$), which is the case of interest in this paper, we obtain a traceless stress-energy tensor.

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